

# Using Edge-induced and Vertex-induced Subhypergraph Polynomials

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## Abstract

For a hypergraph  $\mathcal{H}$ , we consider the edge-induced and vertex-induced subhypergraph polynomials and study their relation. We use this relation to prove that both polynomials are reconstructible, and to prove a theorem relating the Hilbert series of the Stanley-Reisner ring of the independent complex of  $\mathcal{H}$  and the edge-induced subhypergraph polynomial. We also consider reconstruction of some algebraic invariants of  $\mathcal{H}$ .

*Key words and phrases:* edge-induced subhypergraph polynomial, vertex-induced subhypergraph polynomial, Stanley-Reisner ring, Hilbert Series, Reconstruction conjecture.

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## 1 Introduction

To every hypergraph  $\mathcal{H}$  one can associate several subhypergraph enumerating polynomials. In this note we consider two of these polynomials: the vertex-induced subhypergraph polynomial  $P_{\mathcal{H}}(x, y)$  enumerating vertex-induced subhypergraphs of  $\mathcal{H}$ , and the edge-induced subhypergraph polynomial  $S_{\mathcal{H}}(x, y)$ . Precise definitions will be given in §2. These and several other polynomials were extensively studied for graphs, see [1, 4, 5, 8] and their citations. The notion has been naturally generalized to hypergraphs, see White [14].

L. Borzacchini, et al. [5] studied the relation between these and other subgraph enumerating polynomials. He earlier proved that both are reconstructible, i.e. they can be derived from the subgraph enumerating polynomials of vertex-deleted subgraphs, see [3, 4]. A. Goodarzi [9] used  $S_{\mathcal{H}}(x, y)$  to compute the Hilbert series of the Stanley-Reisner ring of the independent complex of  $\mathcal{H}$ . More precisely, if  $R$  is such a ring, then its Hilbert series  $H_R(t)$  is given by

$$(1.1) \quad H_R(t) = \frac{S_{\mathcal{H}}(t, -1)}{(1-t)^n}$$

where  $n$  is the number of vertices in  $\mathcal{H}$ .

In section 2, we define the polynomials, and then prove that

$$S_{\mathcal{H}}(x, y) = (1-x)^n P_{\mathcal{H}}\left(\frac{x}{1-x}, 1+y\right).$$

In section 3, we use this relation to give a short and elementary proof of (1.1). One may compare our proof with the technical proof in [9]. In section 4, generalizing Borzacchini's results [3, 4], we prove that both polynomials are reconstructible for hypergraphs. We also prove the reconstruction problems of some algebraic invariants of the independent complex of  $\mathcal{H}$ , where their graph counter part is proven by Dalili, Faridi and Traves in [6]. That is, we consider reconstructibility of the Hilbert series, the  $f$ -vector, the (multi-)graded Betti numbers and some graded Betti tables of the independent complex of  $\mathcal{H}$ .

## 2 Preliminaries

A hypergraph is a pair  $\mathcal{H} = (V, E)$  where  $V$  is a set of elements called vertices and  $E \subset 2^V$  is a set of distinct subsets of  $V$  called edges such that for any two edges  $\varepsilon_1, \varepsilon_2 \in E$ , we have  $\varepsilon_1 \subset \varepsilon_2 \Rightarrow \varepsilon_1 = \varepsilon_2$ . A hypergraph  $\mathcal{H}$  is called finite if the vertex set  $V$  is finite. We say  $\mathcal{H}$  is a  $d$ -hypergraph if  $|\varepsilon| = d$  for each  $\varepsilon \in E$ , where  $|\varepsilon|$  is the cardinality of  $\varepsilon$ . A graph is a 2-hypergraph. In this note we always consider finite hypergraphs.

Let  $\mathcal{H} = (V, E)$  be hypergraph,  $W \subset V$  and  $L \subset E$ . We say that  $\mathcal{L} = (W, L)$  an *edge-induced subhypergraph* of  $\mathcal{H}$  if  $W = \cup_{\varepsilon \in L} \varepsilon$ . We say that  $\mathcal{H}_W = (W, L)$  is *vertex-induced subhypergraph* if  $L$  is the largest subset of  $E$  such that  $L \subset 2^W$ .

Let  $\mathcal{H}$  be a hypergraph. The *edge-induced subhypergraph polynomial*  $S_{\mathcal{H}}(x, y)$  is defined by

$$(2.1) \quad S_{\mathcal{H}}(x, y) = \sum_{i,j} \theta_{ij} x^i y^j,$$

where  $\theta_{00} = 1$  and for  $i, j \geq 0$ ,  $\theta_{ij}$  is the number of edge induced subhypergraphs of  $\mathcal{H}$  with  $i$  vertices and  $j$  edges. Similarly, the *vertex-induced subhypergraph polynomial*  $P_{\mathcal{H}}(x, y)$  of  $\mathcal{H}$  is defined by

$$(2.2) \quad P_{\mathcal{H}}(x, y) = \sum_{i,j} \beta_{ij} x^i y^j,$$

where  $\beta_{00} = 1$  and for  $i, j \geq 0$ ,  $\beta_{ij}$  is the number of vertex induced subhypergraphs of  $\mathcal{H}$  with  $i$  vertices and  $j$  edges.

We recall some simple properties of these polynomials. In what follows,  $F_{\mathcal{H}}(x, y)$  refers to any one of the two polynomials.

1. If the hypergraph has connected components  $\mathcal{H}_1, \dots, \mathcal{H}_m$ , we have  $F_{\mathcal{H}}(x, y) = F_{\mathcal{H}_1}(x, y) \cdots F_{\mathcal{H}_m}(x, y)$ . We also have  $F(0, y) = 1$ . If  $E = \emptyset$ , then  $F_{\mathcal{H}}(x, y) = (1 + x)^n$ .
2.  $\sum_j \beta_{ij} = \binom{n}{i}$  and  $\sum_i \theta_{ij} = \binom{m}{j}$  where  $m$  is the number of edges in  $\mathcal{H}$ .
3.  $S_{\mathcal{H}}(x, 0)$  is a subgraph polynomial of the 0-subhypergraphs, i.e. isolated vertices.  $P_{\mathcal{H}}(x, 0)$  the polynomial of the independent subsets, i.e. sets of vertices having no edges in common.
4. If  $\mathcal{H} = K_n$  is the complete graph, then  $P_{\mathcal{H}}(x, y) = \sum_{i=0}^n \binom{n}{i} x^i y^{\binom{i}{2}}$  and if  $\mathcal{H}$  is a star with  $m$  edges, then  $S_{\mathcal{H}}(x, y) = \sum_{j=0}^m \binom{m}{j} x^{j+1} y^j$ .

The following Proposition is a generalization of Borzacchini [3]. Even though he considered graphs, the proofs can easily be generalized to hypergraphs.

**Proposition 2.1.** *Let  $\mathcal{H}$  be a hypergraph on  $n$  vertices. Then*

$$S_{\mathcal{H}}(x, y) = (1 - x)^n P_{\mathcal{H}}\left(\frac{x}{1 - x}, 1 + y\right)$$

*Proof.* To every vertex induced subhypergraph with  $i$  vertices and  $l$  edges there are  $\binom{l}{j}$  hypergraphs with  $i$  vertices and  $j$  edges. Moreover, those obtained from different vertex induced subhypergraphs are different since they contain different vertex sets. On the other hand, to every edge induced subhypergraph with  $l$  vertices and  $j$  edges we can construct  $\binom{n-l}{i-j}$  hypergraphs with  $i$  vertices and  $j$  edges. So

$$(2.3) \quad \sum_{l=0}^i \beta_{i,j+l} \binom{j+l}{j} = \sum_{l=0}^i \theta_{i-l,j} \binom{n-(i-l)}{l}.$$

Setting  $r = j + l$  and  $s = i - l$ , substituting this in (2.3) and multiplying it by  $x^i y^j$ , we obtain:

$$\begin{aligned} \sum_{i,j} x^i y^j \left[ \sum_{l=0}^i \beta_{i,j+l} \binom{j+l}{j} \right] &= \sum_{i,j} x^i y^j \left[ \sum_{l=0}^i \theta_{i-l,j} \binom{n-(i-l)}{l} \right]. \\ \sum_{i,j} x^i y^j \left[ \sum_r \beta_{ir} \binom{r}{j} \right] &= \sum_{s,l,j} x^{s+l} y^j \left[ \sum_{l=0}^i \theta_{sj} \binom{n-s}{l} \right]. \\ \sum_{i,r} \beta_{ir} x^i \left[ \sum_j \binom{r}{j} y^j \right] &= \sum_{s,j} \theta_{sj} x^s y^j \left[ \sum_l x^j \binom{n-s}{l} \right]. \\ \sum_{i,r} \beta_{ir} x^i (1 + y)^r &= \sum_{s,j} \theta_{sj} x^s y^j (1 + x)^{n-s}. \\ P_{\mathcal{H}}(x, y + 1) &= (1 + x)^n \sum_{s,j} \theta_{sj} \left( \frac{x}{1 + x} \right)^s y^j. \\ P_{\mathcal{H}}(x, y + 1) &= (1 + x)^n S_{\mathcal{H}}\left(\frac{x}{1 + x}, y\right). \end{aligned}$$

By change of variable, we obtain  $S_{\mathcal{H}}(x, y) = (1 - x)^n P_{\mathcal{H}}\left(\frac{x}{1 - x}, 1 + y\right)$ .  $\square$

**Corollary 2.2.** *Let  $\mathcal{H}$  be a hypergraph on  $n$  vertices. Then*

$$P_{\mathcal{H}}(x, y) = (1 + x)^n S_{\mathcal{H}}\left(\frac{x}{1 + x}, y - 1\right).$$

### 3 $P_{\mathcal{H}}(x, y)$ and $S_{\mathcal{H}}(x, y)$ in Algebra

A *simplicial complex*  $\Delta$  on a vertex set  $V = \{v_1, \dots, v_n\}$  is a set of subsets of  $V$ , called faces or simplices such that  $\{v_i\} \in \Delta$  for each  $i$  and every subset of a face is itself a face. If  $B \subset V$ , the restriction of  $\Delta$  to  $B$  is a simplicial complex defined by  $\Delta(B) = \{\delta \in \Delta \mid \delta \subset B\}$ . The dimension of a face  $\delta \in \Delta$  is  $|\delta| - 1$ . Let  $f_i = f_i(\Delta)$  denote the number of faces of  $\Delta$  of dimension  $i$ . Setting  $f_{-1} = 1$ , the sequence  $f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_{d-1})$  is called the  $f$ -vector of  $\Delta$ .

Let  $A = \mathbb{K}[x_1, \dots, x_n]$  be a polynomial ring over a field  $\mathbb{K}$  and  $\Delta$  be a simplicial complex over  $n$  vertices  $V = \{v_1, \dots, v_n\}$ . The Stanley Reisner ideal of  $\Delta$  is the ideal  $I(\Delta) \subset A$  generated by those square free monomials  $x_{i_1} \cdots x_{i_m}$  where  $\{v_{i_1}, \dots, v_{i_m}\} \notin \Delta$ .

Let  $\mathcal{H} = (V, E)$  be a hypergraph with  $n$  vertices  $V = \{v_1, \dots, v_n\}$ . An independent set of  $\mathcal{H}$  is a subset  $W \subset V$  such that  $\varepsilon \not\subset W$  for all  $\varepsilon \in E$ . The collection of  $\Delta_{\mathcal{H}}$  of independent sets forms a simplicial complex, called the *independent complex*. Thus the Stanley Resiner ideal of  $\Delta_{\mathcal{H}}$  is the edge ideal of  $\mathcal{H}$ . More precisely,  $I(\Delta_{\mathcal{H}}) = I(\mathcal{H}) \subset A$  is the ideal generated by the squarefree monomials  $\prod_{x \in \varepsilon} x$  where  $\varepsilon \in E$ . Conversely, every squarefree monomial ideal  $I \subset A$  can be associated with a hypergraph  $\mathcal{H}_I = (V, E)$  where  $V = \{v_1, \dots, v_n\}$  and  $\varepsilon \in E$  if  $\prod_{x \in \varepsilon} x$  is in the minimal generating set of  $I$ . So one has  $I(\Delta_{\mathcal{H}_I}) = I$ . We have the following lemma.

**Lemma 3.1.** *Let  $(f_0, f_1, \dots, f_{d-1})$  be the  $f$ -vector of the independent complex of a hypergraph  $\mathcal{H}$ . Then  $P_{\mathcal{H}}(t, 0) = \sum_{i=0}^d f_{i-1} t^i$ .*

Let  $R = \bigoplus_{i \in \mathbb{N}} R_n$  be a finitely generated graded  $\mathbb{K}$ -algebra, where  $R_0 = \mathbb{K}$  is a field. The Hilbert series of  $R$  is the generating function defined by  $H_R(t) = \sum_{i \in \mathbb{N}} \dim_{\mathbb{K}}(R_i) t^i$ . If  $I \subset A$  is a monomial ideal, the Hilbert series of the monomial ring  $R = A/I$  is the rational function  $H_R(t) = \frac{\mathcal{K}(R, t)}{(1-t)^n}$  where  $\mathcal{K}(R, t) \in \mathbb{Z}[t]$ . P. Renteln [13], and also D. Ferrarello and R. Fröberg [7] used the subgraph induced polynomial  $S_G(x, y)$  of a graph  $G$  to compute the Hilbert series of the Stanley-Reisner ring  $R$  of the independent complex of  $G$ , namely:

$$H_R(t) = \frac{S_G(t, -1)}{(1-t)^n}.$$

Recently A. Goodarzi [9] generalized it for any squarefree monomial ideal by using the combinatorial Alexander duality and Hochster's formula. Below is a very short and direct proof of this result.

**Theorem 3.2.** *Let  $\mathcal{H}$  be a hypergraph on  $n$  vertices,  $I_{\mathcal{H}} \subset A = \mathbb{K}[x_1, \dots, x_n]$  be its associated squarefree monomial ideal, and  $R = A/I_{\mathcal{H}}$ . Then*

$$H_R(t) = \frac{S_{\mathcal{H}}(t, -1)}{(1-t)^n}.$$

*Proof.* We know by Lemma 3.1 that  $P_{\mathcal{H}}(t, 0) = \sum_{i=0}^d f_{i-1} x^i$  is the polynomial of the  $f$ -vectors of the independent complex of  $\mathcal{H}$ . It follows that by [12, Proposition 51.3] that  $H_R(t) = P_{\mathcal{H}}(\frac{t}{1-t}, 0)$  and by Theorem 2.1 we have

$$S_{\mathcal{H}}(t, -1) = (1-t)^n P_{\mathcal{H}}(\frac{t}{1-t}, 0) = H_R(t)(1-t)^n.$$

□

**Remark 3.3.** Let  $\mathcal{H}$  be a hypergraph and  $R = A/I_{\mathcal{H}}$ . It then follows by Lemma 3.1 and [12, Proposition 51.2] that  $P_{\mathcal{H}}(t, 0)$  is the Hilbert polynomial of the exterior algebra  $R/(x_1^2, \dots, x_n^2)$ .

## 4 $P_{\mathcal{H}}(x, y)$ and $S_{\mathcal{H}}(x, y)$ in reconstruction conjecture

For a graph  $G = (V, E)$  on a vertex set  $V = \{v_1, \dots, v_n\}$ , the deck of  $G$  is the collection  $\mathcal{D}(G) = \{G_1, \dots, G_n\}$  where  $G_l = G - v_l$ ,  $v_l \in V$  is the vertex deleted subgraph of  $G$ . An element of  $\mathcal{D}(G)$  is called a card. The long standing graph reconstruction conjecture posed by Kelly and Ulam says that every simple graph on  $n \geq 3$  vertices is uniquely determined, up to isomorphism, by its deck. Numerous unsuccessful attempts have been made to prove the conjecture, nevertheless, a significant amount of work has been made. The reader may see Bondy [2] for a survey on the subject. Reconstruction of hypergraphs is defined similarly to graphs. Kocay [10] and Kocay and Lui [11] have constructed a family of non-reconstructible 3-hypergraphs.

**Remark 4.1.** Another obvious example of non-constructible hypergraphs are the 0-hypergraph containing no edges, and the  $n$ -hypergraph containing one edge with  $n$ -elements. So all the hypergraphs under consideration in this section are neither of these two.

In recent years questions has been asked if a graph invariant is reconstructible, that is, if it can be obtained from the its deck. Borzacchini in [3, 4] proved that both  $S_G(x, y)$  and  $P_G(x, y)$  are reconstructible. In fact, he proved that if  $F_G(x, y)$  is any one of the subgraph polynomials and  $F_{G_l}(x, y)$  is a subgraph polynomial of the card  $G_l$ , then

$$(4.1) \quad nF_G(x, y) = x \frac{\partial F_G(x, y)}{\partial x} + \sum_{l=1}^n F_{G_l}(x, y).$$

It is natural to extend this reconstructibility question to hypergraphs. Below we obtain a similar result.

**Proposition 4.2.** *Let  $\mathcal{H}$  be a hypergraph on  $n \geq 3$  vertices. Then both  $S_{\mathcal{H}}(x, y)$  and  $P_{\mathcal{H}}(x, y)$  are reconstructible.*

*Proof.* We prove the proposition for  $S_{\mathcal{H}}(x, y)$  since the other will follow by Proposition 2.1. Let  $S_{\mathcal{H}}(x, y) = \sum_{ij} \theta_{ij} x^i y^j$  and  $S_{\mathcal{H}_l}(x, y) = \sum_{ij} \theta_{ij}^{(l)} x^i y^j$  for  $l = 1, \dots, n$ . By direct calculation we have

$$nS_{\mathcal{H}}(x, y) - x \frac{\partial(S_{\mathcal{H}}(x, y))}{\partial x} = n + \sum_{l=1}^n \sum_{ij} (n - j) \theta_{ij} x^i y^j.$$

Now if  $j < n$ , then any edge induced subhypergraph with  $i$  vertices and  $j$  edges is an edge induced subhypergraph for  $n - j$  cards. It follows that  $\sum_{l=1}^n \theta_{ij}^{(l)} = (n - j) \theta_{ij}$ . Putting this in the equation and recalling that  $n = \sum_{l=1}^n \theta_{00}^{(l)}$  we obtain

$$(4.2) \quad nS_{\mathcal{H}}(x, y) = x \frac{\partial S_{\mathcal{H}}(x, y)}{\partial x} + \sum_{i=1}^n S_{\mathcal{H}_i}(x, y).$$

□

## 4.1 Hilbert series and $f$ -vector

The authors in [6] studied reconstructibility of some algebraic invariants of the edge ideal of a graph  $G$  such as the Krull dimension, the Hilbert series, and the graded Betti numbers  $b_{i,j}$ , where  $j < n$ . We extend these results to hypergraphs.

**Proposition 4.3.** *Let  $\mathcal{H}$  be a hypergraph on  $n \geq 3$  vertices. The Hilbert function of  $R = A/I_{\mathcal{H}}$  is reconstructible.*

*Proof.* By Proposition 3.2 and (4.2) we have

$$\begin{aligned} nH_R(t) &= \frac{nS_{\mathcal{H}}(t, -1)}{(1-t)^n} = \frac{t \frac{dS_{\mathcal{H}}(t, -1)}{dt}}{(1-t)^n} + \sum_{i=1}^n \frac{S_{\mathcal{H}_i}(t, -1)}{(1-t)^n} \\ &= \frac{t}{(1-t)^n} \frac{dS_{\mathcal{H}}(t, -1)}{dt} + \sum_{i=1}^n \frac{H_{R_i}(t)}{1-t}. \end{aligned}$$

Since  $\frac{dH_R(t)}{dt} = \frac{d}{dt} \left( \frac{S_{\mathcal{H}}(t, -1)}{(1-t)^n} \right) = \frac{1}{t} \frac{t}{(1-t)^n} \frac{dS_{\mathcal{H}}(t, -1)}{dt} + \frac{n}{1-t} H_R(t)$ , substituting this into the above, we obtain a first order ordinary linear differential equation

$$\frac{n}{1-t} H_R(t) = t \frac{dH_R(t)}{dt} - \frac{1}{1-t} \sum_{i=1}^n H_{R_i}(t).$$

□

**Proposition 4.4.** *Let  $\mathcal{H}$  be a hypergraph on  $n \geq 3$  vertices. The  $f$ -vector of  $\Delta_{\mathcal{H}}$  is reconstructible.*

*Proof.* This, in fact, follows from Proposition 4.3, but we give an independent proof. Let  $f(\Delta_{\mathcal{H}}) = (f_0, \dots, f_{d-1})$ . If  $d < n$ , by (4.2) when  $F = P_{\mathcal{H}}$  and Lemma 3.1, we have  $nf_{l-1} = if_{l-1} + \sum_{i=1}^n f_{i-1}^l$  for all  $l \leq d$ . If  $d = n$ , then  $\mathcal{H}$  has no edges so  $f_{d-1} = 1$ . □

Let  $\Delta_{\mathcal{H}}$  be the independent complex of a hypergraph  $\mathcal{H}$ . We can compute other invariants of  $\Delta_{\mathcal{H}}$  from its  $f$ -vector  $f(\Delta_{\mathcal{H}}) = (f_{-1}, f_0, \dots, f_{d-1})$ . Recall, for example, that the  $h$ -vector  $h(\Delta_{\mathcal{H}}) = (h_0, \dots, h_d)$  is defined by the formula  $\sum_{i=0}^d f_{i-1} t^i (1-t)^{d-i} = \sum_{i=0}^d h_i t^i$ . We can also obtain the multiplicity of the  $R = A/I_{\mathcal{H}}$ , namely  $e(R) = f_{d-1}$ . The following are consequences of Propositions 4.3 and 4.4.

**Corollary 4.5.** *Let  $\mathcal{H}$  be a hypergraph on  $n \geq 3$  vertices. The  $h$ -vector of  $\Delta_{\mathcal{H}}$  is reconstructible.*

**Corollary 4.6.** *Let  $\mathcal{H}$  be a hypergraph on  $n \geq 3$  vertices. Then the Krull dimension and the multiplicity of  $R = A/I_{\mathcal{H}}$  are reconstructible.*

## 4.2 Multi-graded Betti numbers

In this subsection we assume that  $\text{char } \mathbb{K} = 0$ . Let  $I \subset A = \mathbb{K}[x_1, \dots, x_n]$  be a monomial ideal and consider the  $\mathbb{Z}^n$ -graded minimal free resolution of the  $A$ -module  $R = A/I$ :

$$\cdots \rightarrow \oplus_j A(-\mathbf{b})^{b_{i,\mathbf{b}}} \rightarrow \cdots \rightarrow \oplus_j A(-\mathbf{b})^{b_{2,\mathbf{b}}} \rightarrow \oplus_j A(-\mathbf{b})^{b_{1,\mathbf{b}}} \rightarrow A \rightarrow A/I \rightarrow 0$$

where  $\mathbf{b} \in \mathbb{Z}^n$  and the modules  $A(-\mathbf{b})$  are the shifts of  $A$  to make the multi-graded differentials degree zero maps. The numbers  $b_{i,\mathbf{b}}$  are *multi-graded Betti numbers* and  $b_{ij} = \sum_{|\mathbf{b}|=j} b_{i,\mathbf{b}}$ , where  $|\mathbf{b}| = b_1 + \dots + b_n$ , are the *graded Betti numbers* of  $R$ . In particular, the  $b_{in}$ 's are the *extremal graded Betti numbers*. The importance of the assumption that  $\text{char } \mathbb{K} = 0$  is that these numbers depend on the characteristic of the ground field, see eg. [12, Example 12.4]. If  $I = I_{\mathcal{H}}$  is the edge ideal of a hypergraph  $\mathcal{H}$ , then each  $\mathbf{b} \in \{0, 1\}^n$ , see for example [12, Corollary 26.10]. One can use graded Betti numbers to compute the Hilbert series of  $R = A/I_{\mathcal{H}}$ . So by Theorem 3.2, we have

$$(4.3) \quad S_{\mathcal{H}}(t, -1) = \sum_{i=0}^n \sum_j (-1)^i b_{ij} t^j.$$

We generalize [6, Theorem 5.1] with a similar proof.

**Proposition 4.7.** *Let  $\mathcal{H}$  be a hypergraph on with a vertex set  $V = \{v_1, \dots, v_n\}$  and  $n \geq 3$ . Then the multi-graded Betti numbers  $b_{ij}$  of the Stanley Reisner ring  $R = A/I_{\mathcal{H}}$  are reconstructible for all  $j < n$ .*

*Proof.* Let  $\Delta = \Delta_{\mathcal{H}}$ ,  $\Delta^{(l)} = \Delta_{\mathcal{H}_l}$ ,  $\mathbf{b} \in \mathbb{Z}^n$ ,  $b_{i,\mathbf{b}}$  be the multi-graded Betti numbers of  $\Delta$ , and  $b_{i,\mathbf{b}}^{(l)}$  be the multi-graded Betti numbers of  $\Delta^{(l)}$ . By Hochester's formula, we have

$$b_{i,\mathbf{b}} = b_{i,B} = \tilde{h}_{j-i-1}(\Delta(B)),$$

where  $B = \{v_i \in V \mid b_i \neq 0\}$  and  $\tilde{h}_{j-i-1}(\Delta(B)) = \dim_{\mathbb{K}}(\tilde{H}_{j-i-1}(\Delta(B); \mathbb{K}))$  is the reduced simplicial homology of the subcomplex  $\Delta(B)$ . Since  $\Delta(B) = \Delta^{(l)}(B)$  whenever  $v_l \notin B$ , it follows by Hochester's formula that  $b_{i,\mathbf{b}} = \tilde{h}_{j-i-1}(\Delta^{(l)}(B)) = b_{i,\mathbf{b}}^{(l)}$ . So the result holds.  $\square$

**Corollary 4.8.** *Let  $\mathcal{H}$  be a hypergraph with a vertex set  $V = \{v_1, \dots, v_n\}$  and  $n \geq 3$ . Then the graded Betti numbers  $b_{ij}$  of the Stanley Reisner ring  $R = A/I_{\mathcal{H}}$  are reconstructible for all  $j < n$ .*

*Proof.*  $b_{ij} = \sum_{|\mathbf{b}|=j} b_{i,\mathbf{b}}$  and multi-graded Betti numbers are reconstructible.  $\square$

Reconstruction of the extremal graded Betti numbers seems a bit hard to determine. We know that the coefficient of  $t^n$  in  $S_{\mathcal{H}}(t, -1)$  is the alternating sum  $\sum_i (-1)^i b_{in}$ . It follows that  $b_{in}$  is reconstructible if there is only one  $i$  such that  $b_{in} \neq 0$ . Fortunately, we have a good class of ideals with this property: for example, edge ideals of complements of chordal graphs, metroidal ideals, ideals with linear quotients and Cohen-Macaulay ideals. However, there are also edge ideals with more than one non-zero extremal graded Betti numbers [6, Example 5.3]. On the other hand, it is a useful invariant since it gives us information on many other invariants of  $I_{\mathcal{H}}$ . The following extends [6, Proposition 5.4] to hypergraphs.

**Proposition 4.9.** *Let  $\mathcal{H}$  be a hypergraph on  $n \geq 3$  vertices. If the graded top degree Betti numbers  $b_{in}$  of  $I_{\mathcal{H}}$  are reconstructible, then the depth, projective dimension and regularity of  $I_{\mathcal{H}}$  are reconstructible.*

We investigate if the Betti table of  $I_{\mathcal{H}}$  is reconstructible. Let  $\mathcal{B} = (b_{ij})$  be the Betti table of  $I_{\mathcal{H}}$  and  $\mathcal{B}_l = (b_{ij}^{(l)})$  be the Betti table of  $I_{\mathcal{H}_l}$ . Then combining (4.2) and (4.3) and comparing the coefficients of  $t^j$  we obtain

$$(n-j) \sum_i (-1)^i b_{ij} = \sum_i (-1)^i \sum_{l=1}^n b_{ij}^{(l)} \quad \text{for } j < n.$$

This equation shows it is difficult to determine each  $b_{ij}$  only from the data  $\{\mathcal{B}_l\}_{l=1}^n$  since anti-diagonals of  $\mathcal{B}$  might contain more than one non-zero entry. We thus have the following which gives a partial answer to [6, Question 5.6].

**Proposition 4.10.** *Let  $\mathcal{H}$  be a hypergraph on  $n \geq 3$  vertices. If each anti-diagonal of the Betti table of  $I_{\mathcal{H}}$  contains at most one non-zero entry, then the Betti table of  $I_{\mathcal{H}}$  is reconstructible.*

In fact, in this case, we can compute the non-zero entries from the coefficients of  $S_{\mathcal{H}}(x, y)$ .

**Proposition 4.11.** *Let  $\mathcal{H}$  be a hypergraph on  $n \geq 3$  vertices and  $S_{\mathcal{H}}(x, y) = \sum_{ij} \theta_{ij} x^i y^j$ . Assume that each anti-diagonal of the Betti table contains at most one non-zero entry  $b_0, b_1, \dots, b_d$ . Then  $b_i = \sum_j \theta_{ij}$ .*

*Proof.* Since  $S_{\mathcal{H}}(t, -1) = \sum_{ij} \theta_{ij} (-1)^j t^i = \sum_{j=0}^n (-1)^j b_i t^i$ ,  $b_i$  is the coefficient of  $t^i$  in  $S_{\mathcal{H}}(t, -1)$ .  $\square$

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